HADWIGER'S CONJECTURE

BY

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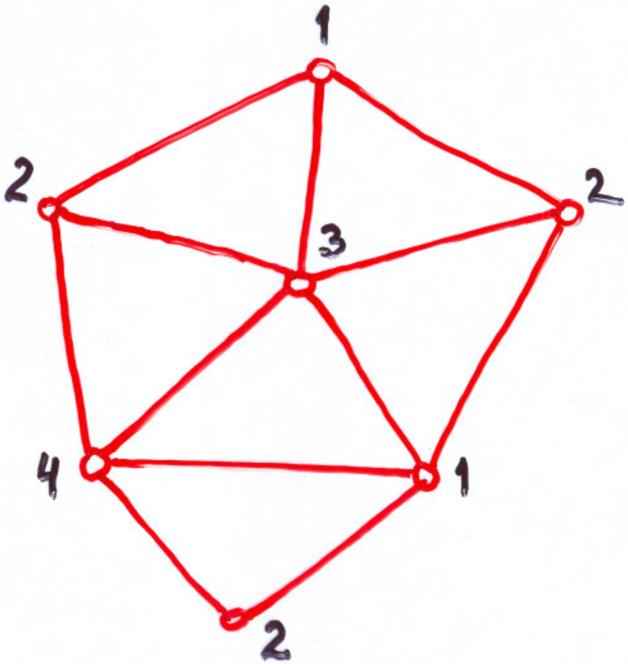
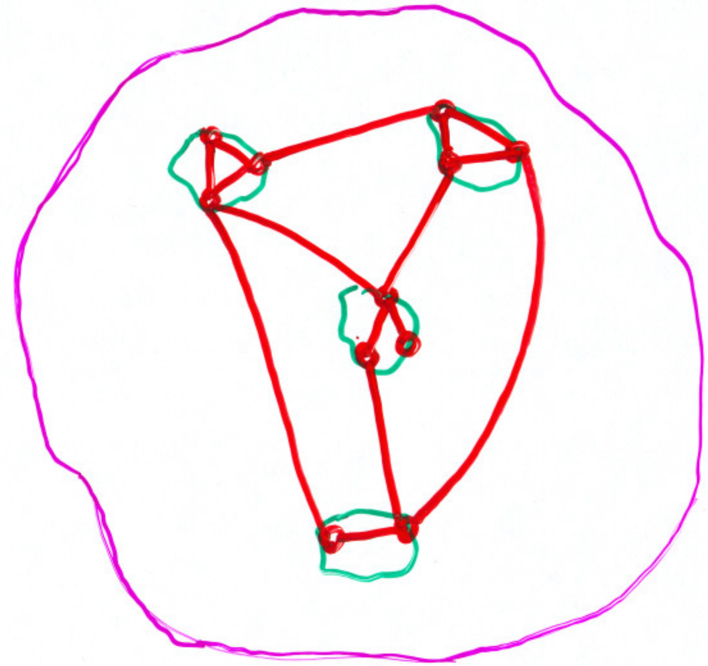
Coast-to-Coast & over 14/3/2006

CHROMATIC NUMBER



$\chi(G) = 4$

K_k - MINOR (substructure)



K_4 as a minor

CONJECTURE (HADWIGER, 1943)

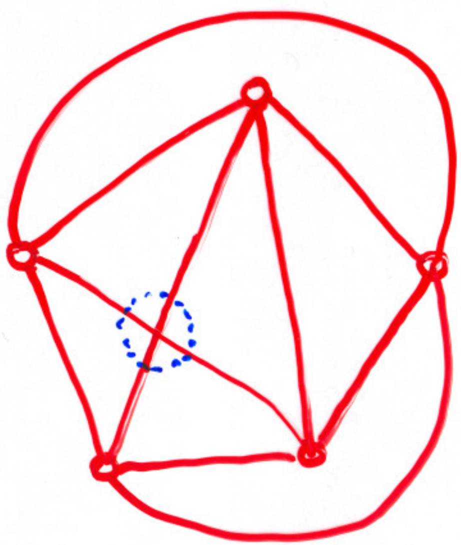
If G is a graph whose chromatic number is equal to k, then G contains the complete graph $\mathbb{K}_k$ as a MINOR.

$$\boxed{\chi(G) \geq k \quad \Longrightarrow \quad K_k\text{-minor}}$$

$$\boxed{\text{No } K_k\text{-minor} \quad \Longrightarrow \quad \chi(G) < k}$$

HADWIGER'S CONJECTURE AS A GENERALIZATION OF THE FOUR-COLOR-THEOREM

FOUR-COLOR-THEOREM: Every planar graph can be 4-colored, $\chi(G) \leq 4$.

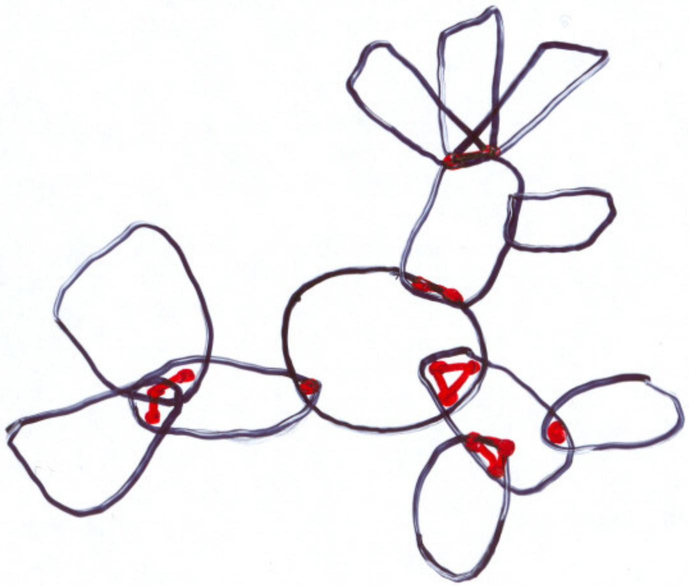


K_5 is non-planar $\Rightarrow$ planar graphs cannot contain K_5-minors $\xrightarrow{(HC)}$ $\chi \leq 4$.

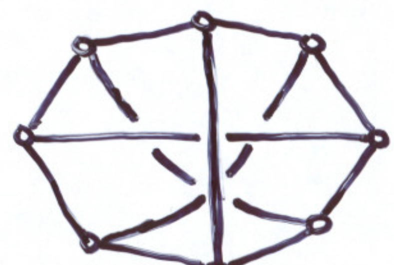
WAGNER'S EQUIVALENCE THEOREM (1937)

$$4CT \Leftrightarrow HC(5).$$

STRUCTURE OF GRAPHS THAT DO NOT CONTAIN K_5-MINORS:



- Tree-like
- small overlap (at most 3 vertices which are "close" -can be made adjacent)
- Building blocks are planar or subgraphs of



V_8

Hadwiger formulated this conjecture as an attempt to generalize 4CC without reference to topology (unlike then known Heawood's Theorem).

Some people believe that HC is today the most outstanding open problem in graph theory (some of us may somewhat disagree).

Althoug the 4CT may not have striking consequences to Mathematics, HC would be much more influential. Its validity would increase our overall understanding of various related problems.

WHAT IS KNOWN

$k \leq 2$	trivial
$k = 3$	almost trivial
$k = 4$	easy (Hadwiger - Dirac)
$k = 5$	4CT (Wagner's Equivalence Theorem)
$k = 6$	4CT (Robertson, Seymour & Thomas) 1993
$k \geq 7$	OPEN

$\boxed{HC(6)}$

THEOREM:
(Robertson, Seymour, and Thomas, 1993)

Every minimal counterexample to HC(6) contains a vertex such that after deleting that vertex a planar graph is obtained.

$$\boxed{HC(6) \Leftrightarrow 4CT}$$

4CT was proved in ~ 1976 by Appel and Haken.
Another proof by RSST (1996).

COLOR-CRITICAL GRAPHS

A minimal counterexample to $HC(k)$ would be **k-CRITICAL**, i.e. $\chi(G) = k$ but every proper subgraph H would have $\chi(H) < k$.

Easy fact: If G is k-critical, then every vertex of G has degree at least $k-1$.

THEOREM (Kostochka, Thomason, 1980's)

There exists a constant c such that every graph with minimum degree at least

$$ck\sqrt{\log k}$$

contains the complete graph K_k as a minor.

COROLLARY (APPROXIMATE H.C.)

If G does not contain K_k-minors, then

$$\chi(G) \leq O(k\sqrt{\log k}).$$

$\alpha = 2$ CONJECTURE

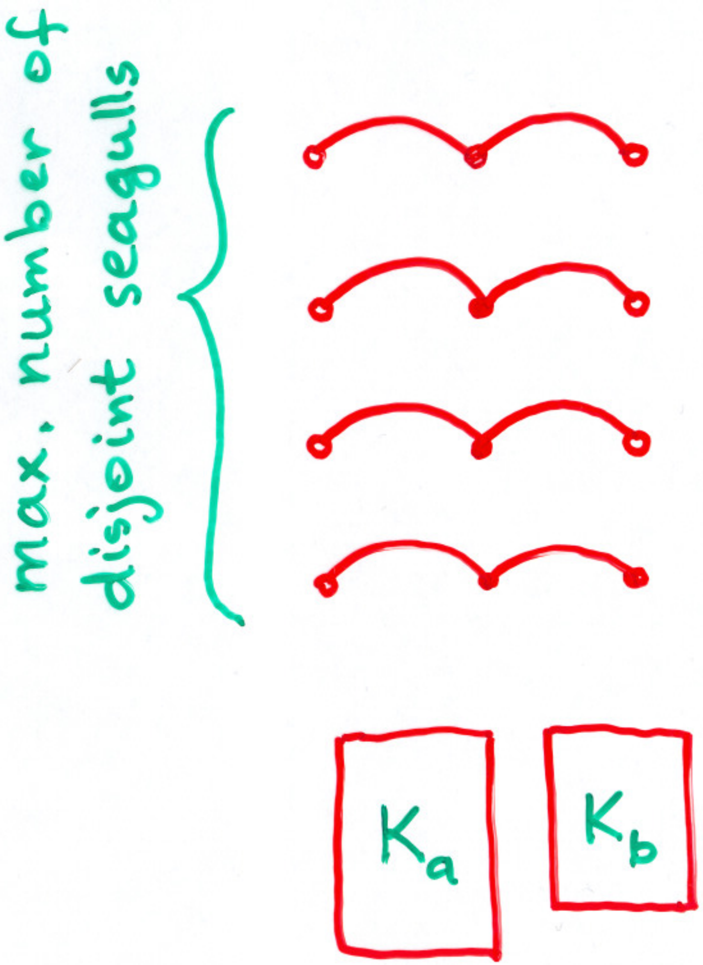
Let G be a graph of order n such that no three vertices in G are mutually non-adjacent. Then $\chi(G) \geq \frac{n}{2}$.

$\alpha(G) :=$ max # of vertices in G that are mutually non-adjacent

$$\chi(G) \geq \frac{n}{\alpha(G)}$$

<u>**CONJECTURE**</u>: If $\alpha(G) = 2$, then G contains $K_{\lceil n/2 \rceil}$-minor.

PROPOSITION: If $\alpha(G) = 2$, then G contains a $K_{\lceil n/3 \rceil}$-minor.



No improvements are known.

The same proof gives a result proved in 1980's by Duchet and Meyniel:

<u>THEOREM</u>. Every graph G satisfies $\chi(G) \geq \frac{|G|}{\alpha(G)}$ and contains K_k-minor for $k = \frac{|G|}{2\alpha(G)-1}$.

Generalized "seagulls"

contains $\leq 2\alpha(G)-1$ vertices

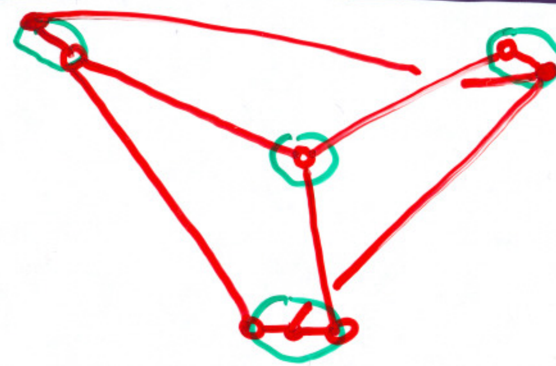
Every other vertex (in other generalized seagulls) is adjacent to some vertex of S.

SOME RELATED CONJECTURES

STRENGTHENING (A) BY GERARDS & SEYMOUR:

$\boxed{\text{CONJ: } \chi(G) \geq k \Rightarrow G \text{ contains an odd } K_k\text{-minor.}}$

Odd minor:



STRENGTHENING (B) BY KAWARABAYASHI & B.M.:

$\boxed{\text{CONJ: No } K_k\text{-minor in } G \Rightarrow ch(G) \leq k.}$

ch(G) .... list-chromatic number (choice number)

Both of these conjectures try to bias the fact that 2-colorable graphs may contain large complete graph minors.

WEAKER CONJECTURES (C), (D):

CONJECTURE: $\chi(G) = k \Rightarrow K_{t,k-t}$-minor for every t, $1 \le t \le k-1$.

CONJECTURE: $\chi(G) = k \Rightarrow K_{3,k-3}$-minor.

WEAKER CONJECTURE (E):

CONJECTURE: $\exists$ constant $c > 0$ such that every graph G of chromatic number k contains $K_{\lceil ck \rceil}$-minor.

Some recent developments

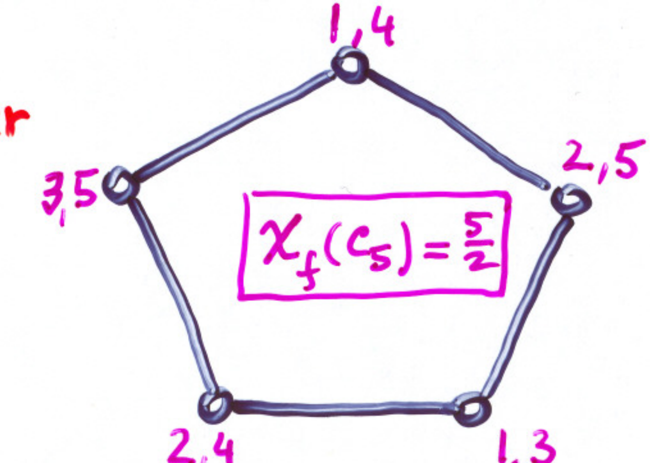
Odd K_k-minors
- Kawarabayashi & Song
- Geelen, Gerards, Goddyn, Reed, Seymour, Vetta

Approximating & list-colorings
- Reed, Seymour
- Kawarabayashi & B.M.

THEOREM (Reed, Seymour): If $\chi_f(G) \geq 2k$, then G contains a K_k-minor.

χ_f .... fractional chromatic number

Each vertex receives t colors, total number of colors is $t \cdot \chi_f$.



APPROXIMATION AND ALGORITHMIC ASPECTS

<u>THEOREM</u> (Kawarabayashi & B.M.):

For every fixed k, there exists a cubic-time algorithm which for a given graph G either

(i) finds a K_k-minor in G, or

(ii) verifies that G is $15.5k$-choosable, or

(iii) finds a subgraph H of G of order $\le N(k)$ s.t. H has no K_k-minors and is not $(9.5k-6)$-choosable.

Alternative result for usual colorings:

- K_k-minor
- $27k$-coloring
- small <u>minor</u> H, no K_k-minors & not $27k$-colorable.

UNPUBLISHED RESULT OF ROBERTSON & SEYMOUR:

If H.C. holds, then

There exists a polynomial-time algorithm for k-coloring graphs in an arbitrary minor-closed family of graphs $\mathcal{M}$ such that $K_k \notin \mathcal{M}$.

- k-coloring, or
- K_{k+1}-minor, or
- small counterex. to H.C.(k).
