COLORINGS OF

GRAPHS ON SURFACES

HISTORY

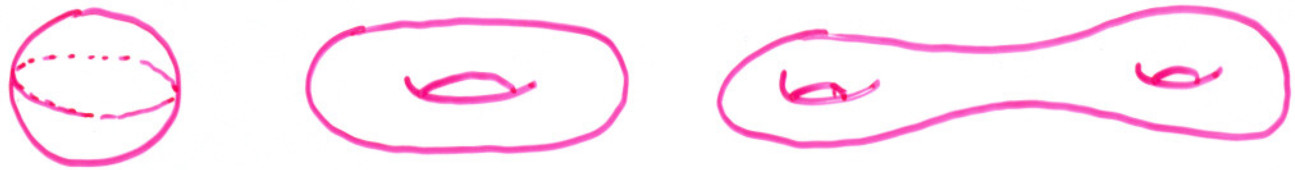
TOOLS

DIRECTIONS

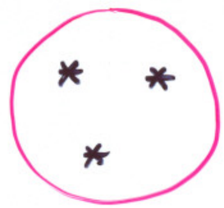
GRAPHS ON SURFACES

CLASSIFICATION OF SURFACES (compact, no boundary)

$\mathbb{S}_g$



$\mathbb{N}_g$



EULER GENUS OF S

$$eg(S) = 2 - \chi(S) = \begin{cases} 2g, & \text{if } S \approx \mathbb{S}_g \\ g, & \text{if } S = \mathbb{N}_g \end{cases}$$

- $|G| - \|G\| + f = \chi(S)$

HEAWOOD THEOREM

Theorem (Heawood, 1890): If a graph G can be embedded in a surface of E. genus g, then

$$\boxed{\chi(G) \leq \left\lfloor \frac{7+\sqrt{1+24g}}{2} \right\rfloor}$$

(when $g=0$, this becomes 4CT).

Dirac proved that the bound is attained if and only if the complete graph of order

$$H(g) := \left\lfloor \frac{7+\sqrt{1+24g}}{2} \right\rfloor$$

can be embedded in the corresponding surface.

Ringel & Youngs (1968): YES for all surfaces except the Klein bottle.

PLANAR GRAPHS

FOUR-COLOR THEOREM: Every planar graph is 4-colorable.

GRÖTZSCH THEOREM: Every triangle-free planar graph is 3-colorable.

ACYCLIC COLORINGS: Every planar graph has an acyclic coloring using at most 5 colors.

A coloring is *acyclic* if every cycle in G receives at least 3 distinct colors.

$\chi_{ac}(G)$

acyclic chromatic number

Grünbaum (9) → Mitchem (8) → Albertson & Berman (7) → → Kostochka (6) → Borodin (5)

ACYCLIC COLORINGS OF GRAPHS OF LARGE GENUS

<u>Conjecture</u> (Borodin): For every surface except the plane, the acyclic chromatic number is bounded above by the Heawood bound $\frac{1}{2}(7+\sqrt{1+24g})$.

Albertson & Berman proved an upper bound $2g+4$.

	Planar graphs	Euler genus g	E.g. g locally planar	Planar graphs List colorings
All graphs	4	$\Theta(g^{1/2})$	5	5
Triangle-free	3	$\tilde{\Theta}(g^{1/3})$	4	4
Acyclic colorings	5	$\tilde{\Theta}(g^{4/7})$	≤ 8	≤ 7

THEOREM (Alon, BM, Sanders): Let G be a graph embedded in a surface of Euler genus g. Then

$$\boxed{\chi_{ac}(G) \leq 100\, g^{4/7} + 10000.}$$

THEOREM: There are graphs of Euler genus g whose acyclic chromatic number is at least

$$\boxed{\chi_{ac}(G) \geq \frac{1}{10}\, g^{4/7} \big/ (\log g)^{1/7}}.$$

In particular, Borodin's conj. fails for almost all surfaces.

BASIC FACTS ABOUT ACYCLIC COLORINGS

(A) Graphs with many edges have large χ_{ac}:

$$\boxed{\chi_{ac}(G) \geq \frac{\|G\|}{|G|} + 1}.$$

<u>Proof</u>. $U_1, \ldots, U_k$ color classes of an acyclic coloring.

$n_i := |U_i|$

$e_{ij} := \|G(U_i \cup U_j)\|$, number of edges joining U_i and U_j.

Acyclic $\Rightarrow$ $e_{ij} \leq n_i + n_j - 1$

$$\|G\| = \frac{1}{2}\sum_i \sum_{j \neq i} e_{ij} \leq \frac{1}{2} \sum_{i=1}^{k} \sum_{\substack{j=1 \\ j \neq i}}^{k} (n_i + n_j - 1) =$$

$$= \frac{1}{2} \sum_i \sum_j n_i + \frac{1}{2} \sum_i \sum_j n_j - \binom{k}{2} = (k-1)|G| - \binom{k}{2}.$$

(B) Graphs with small max. degree have small χ_{ac}:

$$\boxed{\chi_{ac}(G) \leq \lceil 50\, \Delta^{4/3} \rceil}.$$

Proved by Alon, McDiarmid & Reed (1991) by probabilistic techniques.

(C) Let $G \in \mathcal{G}(n,p)$, $p = 3\left(\frac{\log n}{n}\right)^{1/4}$, be a random graph. Then, almost surely,

(i) $\boxed{\|G\| \leq 2n^{7/4}(\log n)^{1/4}}$, and

(ii) $\boxed{\chi_{ac}(G) \geq \frac{n}{2}}$.

$\Rightarrow$ $\chi_{ac}(G) \geq \frac{1}{10} g^{4/7} / (\log g)^{1/7}$

Sketch of pf:

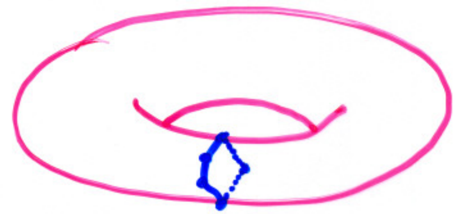
- The number of edges is distributed as a binomial distribution $b(\binom{n}{2}, p)$.
- If $V(G)$ is partitioned in $\leq n/2$ classes $\Rightarrow$ can get $n/4$ pairs of vertices in same classes. For any two such pairs at least one edge between is not present. Probability for that is

$$\leq (1-p^4)^{\binom{n/4}{2}} \leq n^{-2n} \qquad \text{(if } n \text{ large).}$$

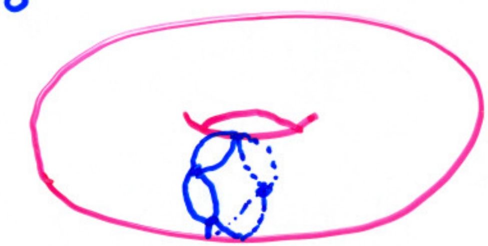
At most n^n partitions into $\leq n/2$ classes........

LOCALLY PLANAR EMBEDDINGS

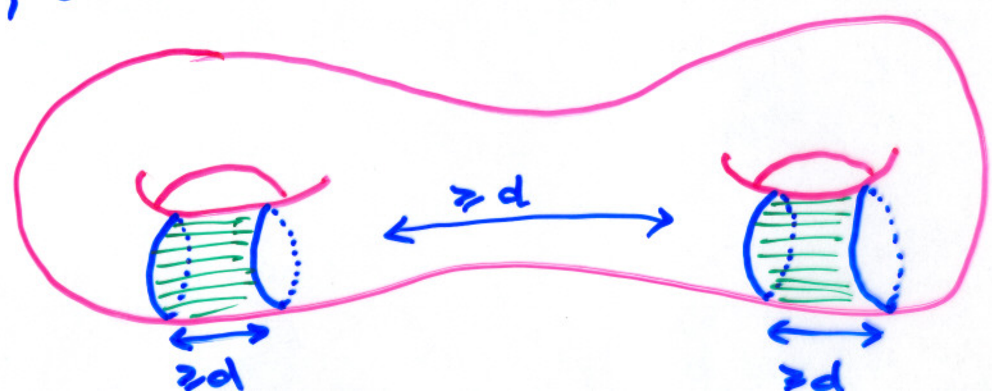
$\boxed{\text{EDGE-WIDTH}}$ $\equiv$ length of a shortest non-contractible cycle of G.



$\boxed{\text{FACE-WIDTH}}$ $\equiv$ smallest # of faces whose union contains a non-contractible cycle.

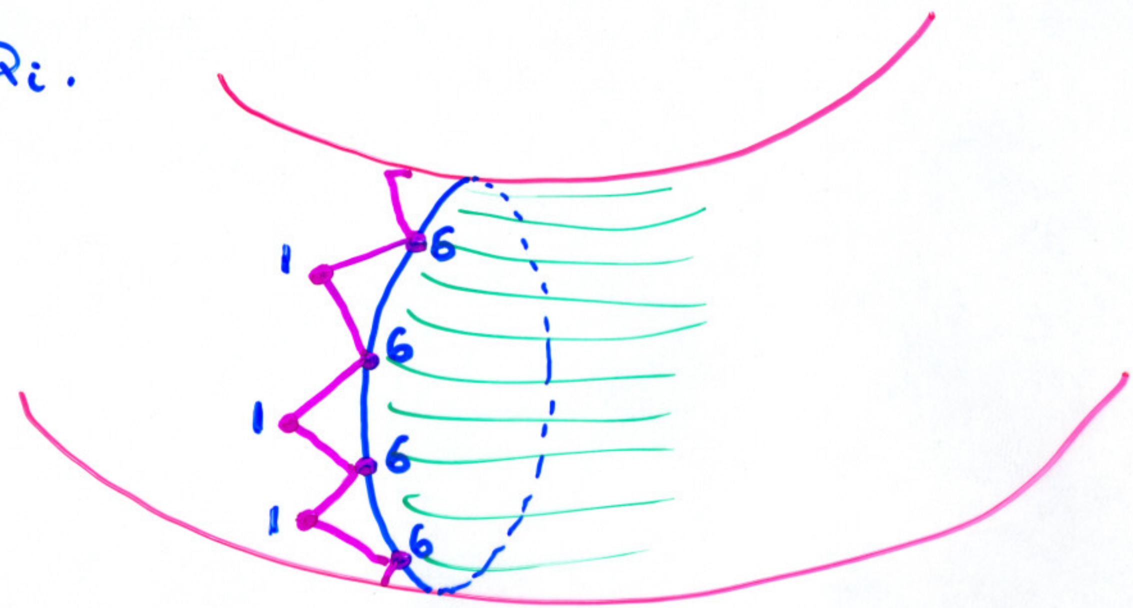


$\boxed{\text{THEOREM}}$ If G is embedded in a surface S and the face-width is "large enough", then G contains a <u>planarizing</u> <u>collection</u> of cycles (cylinders) which are far apart from each other.



- Suppose G has large enough edge-width in S.
- $\exists\, \tilde{G} \supseteq G$ s.t. $\tilde{G}$ has large face-width.
- $\tilde{G}$ contains planarizing cylinders $Q_1, \ldots, Q_{g/2}$ (if S ori.)
- $H := \tilde{G} - (Q_1 \cup \ldots \cup Q_{g/2})$ is planar (use colors $1, 2, \ldots, 5$)
- Each Q_i is planar (use colors $6, 7, \ldots, 10$)
- Combine the colorings to get a coloring of $\tilde{G}$ and
- hence of G.

If a 2-colored cycle arises, it uses a color from H and one from some Q_i.



THEOREM: For every g there exist an integer w s.t. every graph embedded in some surface of Euler genus at most g with edge-width $\geq w$ satisfies:

$$\boxed{\chi_{ac}(G) \leq 8}.$$

Some open problems

PLANAR GRAPHS

$\boxed{P1}$ Characterize critical graphs for acyclic 4-colorings and 5-colorings.

$\boxed{P2}$ Acyclic list colorings with 6 colors.

"HADWIGER CONJECTURE" FOR ACYCLIC CHROMATIC NUMBER

Graphs of genus g with $\chi_{ac}(G) = \tilde{\Theta}(g^{4/7})$ show that these do not have $\Theta(g^{1/2})$-clique minors (surface argument), but not even $\Theta(g^{2/7})$-clique minors because of small number of edges.

$\boxed{P3}$ What is the best constant $q > 0$ such that

$$\mathrm{Hadw}(G) \geq c(\chi_{ac}(G))^{q}.$$

LOCALLY PLANAR EMBEDDINGS

$\boxed{P4}$ Can locally planar graphs on a fixed surface be acyclically colored with 7 (6, 5 ?) colors?

$\boxed{P5}$ What is the worst acyclic chromatic number for locally planar graphs on the projective plane, torus, Klein bottle ?

SMALL WIDTH

$\boxed{P6}$ Acyclic chromatic number for small surfaces ?
(P.P. ≤ 7, ≥ 6 / K.b. ≥ 7 / Torus ?)

ACYCLIC LIST-COLORINGS

$\boxed{P1' - P6'}$ All of the above for list colorings.

$\boxed{P7}$ Acyclic & large girth.